Geometric evolution of the Reynolds stress tensor in three-dimensional turbulence

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Abstract

The dynamics of the Reynolds stress tensor is determined by an evolution equation coupling geometrical effects and turbulent source terms. The effects of the mean flow geometry are shown up when the source terms are neglected. Then, the Reynolds stress tensor is expressed as the sum of three tensor products of vector fields, which are governed by a *distorted gyroscopic* equation. Along the mean flow trajectories and in the directions of the vector fields, the fluctuations of velocity are determined by differential equations whose coefficients depend only on the mean flow deformation.

1 Introduction

The governing equations of barotropic turbulent compressible fluids are (see, for example [1], [2], [3])

$$\left\{\begin{array}{c} \langle\rho\rangle_t + (\langle\rho\rangle\, U_i)_{,i} = 0, \\[2ex] (\langle\rho\rangle\, U_i)_t + (\langle\rho\rangle\; U_iU_j + \langle p\rangle\; \delta_{ij} + \langle\rho\, u_iu_j\rangle)_{,j} = 0, \\[2ex] \langle\rho\, u_iu_j\rangle_t + (\langle\rho\, u_iu_j\rangle\; U_k)_{,k} + \langle\rho\, u_iu_j\rangle\; U_{j,k} + \langle\rho\, u_iu_j\rangle\; U_{i,k} = S_{ij}\,. \end{array}\right. \tag{1}$$

Here "brackets" mean the averaging, "coma" means the derivation with respect to the Eulerian coordinates $\mathbf{x} = \{x_i\},\ i \in \{1,2,3\}$, index t means the partial derivative with respect to time, ρ is the fluid density, $\mathbf{U} = \{U_i\},\ i \in \{1,2,3\}$ is the mass average velocity, p is the pressure, $\mathbf{u} = \{u_i\},\ i \in \{1,2,3\}$ is the velocity fluctuation verifying $\langle\rho\mathbf{u}\rangle = 0$. Repeated indices mean summation. The term $\mathbf{S} = \{S_{ij}\}$ represents turbulent sources. We introduce the Reynolds stress tensor

$$\mathbf{R} = \langle\rho\ \mathbf{u}\otimes\mathbf{u}\rangle\,, \quad (R_{ij} = \langle\rho\, u_iu_j\rangle).$$

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The system (1) can also be rewritten in the tensorial form

$$\begin{cases} \dfrac{\partial \langle \rho \rangle}{\partial t} + \operatorname{div}\left(\langle \rho \rangle \mathbf{U}\right) = 0, \\ \\ \langle \rho \rangle \dfrac{d\mathbf{U}}{dt} + \nabla \langle p \rangle + (\operatorname{div} \mathbf{R})^T = 0, \\ \\ \dfrac{d\mathbf{R}}{dt} + \mathbf{R} \operatorname{div}\mathbf{U} + \dfrac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{R} + \mathbf{R}\left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T = \mathbf{S}, \end{cases} \tag{2}$$

where d/dt means the material derivative with respect to the mean motion

$$\frac{d}{dt} = \frac{\partial}{\partial t} + \mathbf{U}^T \ \nabla \ .$$

The superscript " ^{T} " means the transposition. By using the mass conservation law, the equation for the volumic Reynolds tensor $\mathbf{R}$ can be rewritten as the equation for the specific (or per unit mass) Reynolds tensor

$$\frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{P} + \mathbf{P}\left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T = \frac{\mathbf{S}}{\langle \rho \rangle}, \tag{3}$$

where

$$\mathbf{P} = \frac{\mathbf{R}}{\langle \rho \rangle}.$$

The aim of this note is to understand the structure of homogeneous Reynolds equation (3) corresponding to $\mathbf{S} = \mathbf{0}$. The reason is twofold:
- first, in the numerical study of compressible turbulent flows, this is a natural step in applying the splitting-up technique (see for example [4]),
- second, the system (2) also appears as an exact asymptotic model of weakly shearing flows of long waves over a flat bottom [5]

$$\begin{cases} \dfrac{\partial h}{\partial t} + \operatorname{div}\left(\ h \mathbf{U}\right) = 0, \\ \\ h\dfrac{d\mathbf{U}}{dt} + \nabla\left(\dfrac{gh^2}{2}\right) + (\operatorname{div} \mathbf{R})^T = 0, \\ \\ \dfrac{d\mathbf{R}}{dt} + \mathbf{R} \operatorname{div}\mathbf{U} + \dfrac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{R} + \mathbf{R}\left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T = 0. \end{cases} \tag{4}$$

In Eq. (4), h is the fluid depth, the average pressure is given by $\langle p \rangle = gh^2/2$, g is the gravity acceleration, and

$$\mathbf{R} = \int_0^h (\widetilde{\mathbf{U}} - \mathbf{U}) \otimes (\widetilde{\mathbf{U}} - \mathbf{U}) dz, \qquad h\mathbf{U} = \int_0^h \widetilde{\mathbf{U}} dz,$$

where $\widetilde{\mathbf{U}}$ is an instantaneous velocity. Equations are written for three-dimensional long waves and the production term is zero in the limit of weakly shearing flows. Also, Eqs. (4) are hyperbolic. Finally, we will focus on the equation of the Reynolds tensor per unit mass

$$\frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{P} + \mathbf{P}\left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T = 0. \tag{5}$$

The particular case rot $\mathbf{U} = 0$ was investigated in [6]. In such a case $\left(\partial\mathbf{U}/\partial\mathbf{x}\right)^T = \partial\mathbf{U}/\partial\mathbf{x}$ and Eq. (5) corresponds to a two-covariant tensor convected by the mean flow. This means that $\mathbf{P}$ has a zero Lie derivative d_L with respect to the velocity field $\mathbf{U}$ and the tensor $\mathbf{P}_0$, image of $\mathbf{P}$ in Lagrange coordinates $(t, \mathbf{X})$, only depends on $\mathbf{X} = \{X_i\},\ i \in \{1,2,3\}$

$$d_L\mathbf{P} \equiv \frac{d\mathbf{P}}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{P} + \mathbf{P}\frac{\partial \mathbf{U}}{\partial \mathbf{x}} = 0, \qquad \mathbf{P} = \left(F^T\right)^{-1}\mathbf{P}_0\left(\mathbf{X}\right)F^{-1},$$

where $F = \partial\mathbf{x}/\partial\mathbf{X}$ is the deformation gradient of the mean motion [7].

The aim of the paper is the study of the homogeneous Reynolds tensor equations structure (5) in ***the case*** $\texttt{rot}\,\mathbf{U} \neq 0$.

2 Geometrical properties of the Reynolds tensor evolution

The Reynolds stress tensor $\mathbf{P}$ is symmetric, semi-positive definite and consequently can be rewritten in a local basis of orthonormal eigenvectors in the form

$$\mathbf{P} = \sum_{\alpha=1}^{3} \lambda_\alpha^2\ \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha \equiv \sum_{\alpha=1}^{3} \lambda_\alpha^2\ \mathbf{e}_\alpha\,\mathbf{e}_\alpha^T.$$

The eigenvalues λ_α^2, $\alpha \in \{1,2,3\}$ are non-negative; the case $\lambda_\alpha^2 > 0$ is a generic one. For the two-dimensional case, $\lambda_3^2 \equiv 0$. Let us denote

$$\mathbf{a}_\alpha = \lambda_\alpha\ \mathbf{e}_\alpha, \quad (\lambda_\alpha > 0) \qquad \alpha \in \{1,2,3\}.$$

Then,

$$\mathbf{P} = \sum_{\alpha=1}^{3} \mathbf{a}_\alpha \otimes \mathbf{a}_\alpha \equiv \sum_{\alpha=1}^{3} \mathbf{a}_\alpha\,\mathbf{a}_\alpha^T. \tag{6}$$

From Eq. (6), we deduce

$$\frac{d\mathbf{P}}{dt} = \sum_{\alpha=1}^{3} \frac{d\mathbf{a}_\alpha}{dt}\,\mathbf{a}_\alpha^T + \mathbf{a}_\alpha\left(\frac{d\mathbf{a}_\alpha}{dt}\right)^T. \tag{7}$$

By using Eq. (7), Eq. (5) can be written

$$\sum_{\alpha=1}^{3}\left(\frac{d\mathbf{a}_\alpha}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{a}_\alpha\right)\mathbf{a}_\alpha^T + \left[\left(\frac{d\mathbf{a}_\alpha}{dt} + \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{a}_\alpha\right)\mathbf{a}_\alpha^T\right]^T = 0. \tag{8}$$

The vector $d\mathbf{a}_\alpha/dt+(\partial\mathbf{U}/\partial\mathbf{x})\,\mathbf{a}_\alpha$ can be developped in the local basis $\{\mathbf{a}_\alpha\},\ \alpha\in\{1,2,3\}$ of eigenvectors; one obtains

$$\frac{d\mathbf{a}_\alpha}{dt}+\frac{\partial\mathbf{U}}{\partial\mathbf{x}}\,\mathbf{a}_\alpha=\sum_{\beta=1}^{3}A_{\beta\alpha}\,\mathbf{a}_\beta,\qquad \alpha\in\{1,2,3\} \tag{9}$$

where $A_{\beta\alpha},\ (\alpha,\beta\in\{1,2,3\})$ are the scalar components to be determined. By using Eq. (9), Eq. (8) leads to

$$\sum_{\alpha=1}^{3}A_{\alpha\alpha}\,\mathbf{g}_{\alpha\alpha}+\sum_{\alpha\neq\beta=1}^{3}\left(A_{\alpha\beta}+A_{\beta\alpha}\right)\,\mathbf{g}_{\alpha\beta}=0,$$

where $\mathbf{g}_{\alpha\alpha}=2\mathbf{a}_\alpha\,\mathbf{a}_\alpha^T$ and $\mathbf{g}_{\alpha\beta}=\mathbf{g}_{\beta\alpha}=\mathbf{a}_\alpha\,\mathbf{a}_\beta^T+\mathbf{a}_\beta\,\mathbf{a}_\alpha^T,\ (\alpha,\beta\in\{1,2,3\})$, are six independent symmetric tensors. Consequently,

$$A_{\alpha\alpha}=0\quad\text{and}\quad A_{\alpha\beta}+A_{\beta\alpha}=0,\qquad \alpha,\beta\in\{1,2,3\}.$$

Equation (8) is equivalent to

$$\frac{d\mathbf{a}_\alpha}{dt}+\frac{\partial\mathbf{U}}{\partial\mathbf{x}}\,\mathbf{a}_\alpha=\Lambda\,i\,(\boldsymbol{\pi})\,\mathbf{e}_\alpha\ \ \text{with}\ \ \boldsymbol{\pi}=A_{32}\,\mathbf{e}_1+A_{13}\,\mathbf{e}_2+A_{21}\,\mathbf{e}_3,\quad \alpha\in\{1,2,3\}, \tag{10}$$

where diagonal matrix Λ and antisymmetric matrix $i\,(\boldsymbol{\pi})$ are determined in the basis $\mathbf{e}_\alpha,\ \alpha\in\{1,2,3\}$ as

$$\Lambda=\begin{pmatrix}\lambda_1&0&0\\0&\lambda_2&0\\0&0&\lambda_3\end{pmatrix},\qquad i\,(\boldsymbol{\pi})=\begin{pmatrix}0&-A_{21}&A_{13}\\A_{21}&0&-A_{32}\\-A_{13}&A_{32}&0\end{pmatrix}.$$

The vectors $\mathbf{a}_\alpha,\ \alpha\in\{1,2,3\}$ are orthogonal, $\mathbf{a}_\alpha^T\,\mathbf{a}_\beta=0,\ (\alpha\neq\beta)$. If we assume the orthogonality at initial instant, this is equivalent to

$$\mathbf{a}_\alpha^T\,\frac{d\mathbf{a}_\beta}{dt}+\mathbf{a}_\beta^T\,\frac{d\mathbf{a}_\alpha}{dt}=0. \tag{11}$$

So, Eqs. (10) - (11) yield

$$\forall\,\alpha\neq\beta\in\{1,2,3\},$$

$$\mathbf{a}_\alpha^T\,\left(\Lambda\,i\,(\boldsymbol{\pi})\,\mathbf{e}_\beta-\frac{\partial\mathbf{U}}{\partial\mathbf{x}}\,\mathbf{a}_\beta\right)+\mathbf{a}_\beta^T\,\left(\Lambda\,i\,(\boldsymbol{\pi})\,\mathbf{e}_\alpha-\frac{\partial\mathbf{U}}{\partial\mathbf{x}}\,\mathbf{a}_\alpha\right)=0,$$

or

$$\begin{aligned}2\lambda_\alpha\lambda_\beta\mathbf{e}_\alpha^T\,\mathbf{D}\,\mathbf{e}_\beta&=\mathbf{a}_\alpha^T\,\Lambda\,i\,(\boldsymbol{\pi})\,\mathbf{e}_\beta+\mathbf{a}_\beta^T\,\Lambda\,i\,(\boldsymbol{\pi})\,\mathbf{e}_\alpha\\&\equiv\lambda_\alpha^2\mathbf{e}_\alpha^T\,i\,(\boldsymbol{\pi})\,\mathbf{e}_\beta+\lambda_\beta^2\mathbf{e}_\beta^T\,i\,(\boldsymbol{\pi})\,\mathbf{e}_\alpha\end{aligned} \tag{12}$$

where

$$\mathbf{D} = \frac{1}{2}\left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}} + \left(\frac{\partial \mathbf{U}}{\partial \mathbf{x}}\right)^T\right)$$

is the rate of deformation tensor corresponding to the mean flow. We denote the mixed product of three vectors $\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$ as $(\mathbf{a}, \mathbf{b}, \mathbf{c}) \equiv \mathbf{a}^T (\mathbf{b} \wedge \mathbf{c})$. Hence, Eq. (12) is

$$\begin{aligned} 2\lambda_\alpha \lambda_\beta \, \mathbf{e}_\alpha^T \; \mathbf{D} \; \mathbf{e}_\beta &= \left(\lambda_\alpha^2 - \lambda_\beta^2 \right) \left(\mathbf{e}_\alpha, \boldsymbol{\pi}, \mathbf{e}_\beta \right) \\ &= \left(\lambda_\beta^2 - \lambda_\alpha^2 \right) \left(\boldsymbol{\pi}, \mathbf{e}_\alpha, \mathbf{e}_\beta \right) \\ &= \left(\lambda_\beta^2 - \lambda_\alpha^2 \right) \boldsymbol{\pi}^T \mathbf{e}_\gamma , \end{aligned}$$

where $\{\alpha, \beta, \gamma\}$ is a cyclic permutation of the triplet $\{1, 2, 3\}$. Finally, we get

$$\boldsymbol{\pi}^T \mathbf{e}_\gamma = \frac{2\lambda_\alpha \lambda_\beta \; \mathbf{e}_\alpha^T \; \mathbf{D} \; \mathbf{e}_\beta}{\lambda_\beta^2 - \lambda_\alpha^2}. \tag{13}$$

Equation (10) can be written

$$\lambda_\alpha \frac{d\mathbf{e}_\alpha}{dt} + \frac{d\lambda_\alpha}{dt}\,\mathbf{e}_\alpha + \lambda_\alpha \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{e}_\alpha \; - \Lambda \, i\,(\boldsymbol{\pi})\,\mathbf{e}_\alpha = 0, \qquad \alpha \in \{1, 2, 3\}. \tag{14}$$

Since

$$\mathbf{e}_\alpha^T \frac{d\mathbf{e}_\alpha}{dt} = 0, \qquad \mathbf{e}_\alpha^T \Lambda \, i\,(\boldsymbol{\pi})\,\mathbf{e}_\alpha = \lambda_\alpha \mathbf{e}_\alpha^T i\,(\boldsymbol{\pi})\,\mathbf{e}_\alpha = 0,$$

by multiplying the left side of Eq. (14) with $\mathbf{e}_\alpha^T$, we get

$$\frac{d\lambda_\alpha}{dt} + \; \left(\mathbf{e}_\alpha^T \; \mathbf{D} \; \mathbf{e}_\alpha\right) \lambda_\alpha = 0.$$

By multiplying the left side of Eq. (14) with the projector $\left(\mathbf{I} - \mathbf{e}_\alpha \mathbf{e}_\alpha^T\right)$, we get

$$\lambda_\alpha \frac{d\mathbf{e}_\alpha}{dt} + \lambda_\alpha \left(\mathbf{I} - \mathbf{e}_\alpha \mathbf{e}_\alpha^T\right) \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{e}_\alpha \; - \Lambda \, i\,(\boldsymbol{\pi})\,\mathbf{e}_\alpha = 0.$$

Due to the fact there exists a vector $\mathbf{\Pi}$ such that

$$\frac{d\mathbf{e}_\alpha}{dt} \; = \mathbf{\Pi} \wedge \mathbf{e}_\alpha,$$

vector $\mathbf{\Pi}$ verifies the condition

$$\lambda_\alpha i\,(\mathbf{\Pi})\,\mathbf{e}_\alpha + \lambda_\alpha \left(\mathbf{I} - \mathbf{e}_\alpha \mathbf{e}_\alpha^T\right) \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{e}_\alpha \; - \Lambda \, i\,(\boldsymbol{\pi})\,\mathbf{e}_\alpha = 0. \tag{15}$$

By multiplying Eq. (15) with $\mathbf{e}_\beta^T$ where $\beta \neq \alpha$,we get

$$\lambda_\alpha \mathbf{\Pi}^T \mathbf{e}_\gamma = \lambda_\beta \boldsymbol{\pi}^T \mathbf{e}_\gamma - \lambda_\alpha \mathbf{e}_\beta^T \frac{\partial \mathbf{U}}{\partial \mathbf{x}}\,\mathbf{e}_\alpha. \tag{16}$$

By replacing Rel. (13) into Eq. (16) we get

$$
\begin{aligned}
\mathbf{\Pi}^T \mathbf{e}_\gamma &= \frac{\lambda_\beta^2 \mathbf{e}_\alpha^T \left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}} + \left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T \right) \mathbf{e}_\beta}{\lambda_\beta^2 - \lambda_\alpha^2} - \mathbf{e}_\beta^T \frac{\partial \mathbf{U}}{\partial \mathbf{x}} \mathbf{e}_\alpha \\
&= \frac{\mathbf{e}_\beta^T \left(\lambda_\beta^2 \left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T + \lambda_\alpha^2 \dfrac{\partial \mathbf{U}}{\partial \mathbf{x}} \right) \mathbf{e}_\alpha}{\lambda_\beta^2 - \lambda_\alpha^2}.
\end{aligned}
$$

We can now formulate the following result:

Theorem 1. *The Reynolds tensor can be written in the form*

$$
\mathbf{R} = \langle \rho \rangle \sum_{\alpha=1}^{3} \lambda_\alpha^2 \ \mathbf{e}_\alpha \otimes \mathbf{e}_\alpha.
$$

The eigenvectors $\mathbf{e}_\alpha$ and the eigenvalues λ_α verify the equations:

$$
\left\{
\begin{aligned}
& \frac{d\mathbf{e}_\alpha}{dt} = \mathbf{\Pi} \wedge \mathbf{e}_\alpha, \\
& \frac{d\left(\mathrm{Ln}\, \lambda_\alpha^2\right)}{dt} = -2\ \mu_\alpha,
\end{aligned}
\right.
\qquad \alpha \in \{1,2,3\} \tag{17}
$$

where

$$
\mu_\alpha = \mathbf{e}_\alpha^T\ \mathbf{D}\ \mathbf{e}_\alpha, \qquad \mathbf{\Pi}^T \mathbf{e}_\gamma = \frac{\mathbf{e}_\beta^T \left(\lambda_\beta^2 \left(\dfrac{\partial \mathbf{U}}{\partial \mathbf{x}} \right)^T + \lambda_\alpha^2 \dfrac{\partial \mathbf{U}}{\partial \mathbf{x}} \right) \mathbf{e}_\alpha}{\lambda_\beta^2 - \lambda_\alpha^2}.
$$

The triplet $\{\alpha, \beta, \gamma\}$ corresponds to a cyclic permutation of the triplet $\{1,2,3\}$.

Equation (17_1) is similar to the equations of a rigid body [8]. The vectors $\mathbf{e}_\alpha$ form a natural moving frame $G = \{\mathbf{e}_\alpha\}_{\alpha=1}^3$ depending on the mean deformation. The eigenvalues λ_α^2 of the Reynolds tensor are determined by the evolution equation (17_2). Note that if λ_α are initially positive, they will be positive at any time. Hence, the tensor $\mathbf{P}$ is always positive definite.

Due to the mass conservation law (2_1) and Eq. (17_2), we obtain

$$
\frac{d}{dt} \left(\langle \rho \rangle^{-2} \prod_{\alpha=1}^{3} \lambda_\alpha^2 \right) = 0.
$$

Consequently, system (17) admits a scalar invariant along the trajectories of mean flow. This invariant was earlier obtained in [6] in a different form.

Let us introduce the turbulent specific energy

$$e_T = \frac{1}{2}\,\mathtt{tr}\,\mathbf{P} = \frac{1}{2}\sum_{\alpha=1}^{3}\lambda_\alpha^2.$$

In the incompressible (isochoric) case, we have $d\langle\rho\rangle/dt = 0$; the turbulent energy is minimal in the isotropic case when the three eigenvalues λ_α^2 are equal ($\lambda_1^2 = \lambda_2^2 = \lambda_3^2 = \lambda^2$).
In this case, the orthonormal eigenvectors $\mathbf{e}_\alpha,\ \alpha \in \{1,2,3\}$ of the Reynolds stress tensor $\mathbf{P}$ are also the orthonormal eigenvectors of the mean rate of deformation tensor $\mathbf{D}$. The eigenvalues of $\mathbf{D}$ are μ_α, $\alpha \in \{1,2,3\}$.

In the compressible isotropic case $e_T = \langle\rho\rangle^{2/3}\kappa$, $\kappa = 3\lambda^2/(2\,\langle\rho\rangle^{2/3})$, and κ is a classical invariant of isotropic turbulence. In presence of shock waves the quantity $\langle\rho\rangle^{-2}\prod_{\alpha=1}^{3}\lambda_\alpha^2$ is not conserved through shocks; it increases like the classical entropy in compressible fluid dynamics. The estimation of the jump of turbulence entropy in isotropic case was given in [9].
The governing equations (2) admit the energy conservation law

$$\frac{\partial}{\partial t}\left(\langle\rho\rangle\left(\frac{1}{2}|\mathbf{U}|^2 + e_i + e_T\right)\right) + \mathrm{div}\left(\langle\rho\rangle\mathbf{U}\left(\frac{1}{2}|\mathbf{U}|^2 + e_i\right) + \left(\langle p\rangle\mathbf{I} + \mathbf{R}\right)\mathbf{U}\right) = 0,$$

where the internal specific energy e_i is defined by

$$de_i = -\langle p\rangle d\left(\frac{1}{\langle\rho\rangle}\right)$$

and the mean pressure $\langle p\rangle$ is supposed to be a given function of $\langle\rho\rangle$. Indeed, using (17_2) we immediately obtain

$$\frac{\partial}{\partial t}\left(\langle\rho\rangle\left(\frac{1}{2}|\mathbf{U}|^2 + e_i + e_T\right)\right) + \mathrm{div}\left(\langle\rho\rangle\mathbf{U}\left(\frac{1}{2}|\mathbf{U}|^2 + e_i\right) + \left(\langle p\rangle\mathbf{I} + \mathbf{R}\right)\mathbf{U}\right) =$$

$$\langle\rho\rangle\frac{de_T}{dt} + \mathtt{tr}\,(\mathbf{R}\,\mathbf{D}) =$$

$$\frac{\langle\rho\rangle}{2}\frac{d}{dt}\left(\sum_{\alpha=1}^{3}\lambda_\alpha^2\right) + \langle\rho\rangle\,\mathtt{tr}\left(\sum_{\alpha=1}^{3}\lambda_\alpha^2\mu_\alpha\right) = 0.$$

3 Conclusion

The equations of fluid turbulent motions take three equations into account: the equation of the mass balance (1_1), the balance equation of the average momentum (1_2), and the Reynolds stress tensor equation of evolution (1_3); this last

equation has been the object of our study. If the turbulent sources are neglected, the turbulent fluid motion is a superposition of the mean motion and turbulent fluctuations. The eigenvectors of the Reynolds tensor carry the fluctuations associated with the mean flow deformation. The amplitude of turbulent deformations is defined by the eigenvalues of the Reynolds stress tensor. Locally, the equations for the directions of turbulent fluctuations describe a small solid whose rotation is given by a gyroscopic type equation (Equation (17_1)). The amplitude evolution of turbulent deformations is determined by the diagonal values μ_α of the mean deformation tensor $\mathbf{D}$ expressed in the eigenvector basis of the Reynolds stress tensor. The turbulence increases with the time when $\mu_\alpha < 0$, and decreases when $\mu_\alpha > 0$. In the particular case of incompressible fluid motions we have $\mathtt{tr}\,\mathbf{D} = 0$, and hence there always exists a direction in which the turbulence is increasing while in other directions it is decreasing. These mathematical deductions are confirmed by experiments [10].

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